

CHARACTERIZATIONS AND INTEGRAL FORMULAE FOR GENERALIZED m -QUASI-EINSTEIN METRICS

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ABSTRACT. The aim of this paper is to present some structural equations for generalized m -quasi-Einstein metrics $(M^n, g, \nabla f, \lambda)$, which was defined recently by Catino in [11]. In addition, supposing that M^n is an Einstein manifold we shall show that it is a space form with a well defined potential f . Finally, we shall derive a formula for the Laplacian of its scalar curvature which will give some integral formulae for such a class of compact manifolds that permit to obtain some rigidity results.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In recent years, much attention has been given to classification of Riemannian manifolds admitting an Einstein-like structure, which are natural generalization of the classical Ricci solitons. For instance, Catino in [11] introduced a class of special Riemannian metrics which naturally generalizes the Einstein condition. More precisely, he defined that a complete Riemannian manifold (M^n, g) , $n \geq 2$, is a *generalized quasi-Einstein metric* if there exist three smooth functions f , λ and μ on M , such that

$$(1.1) \quad Ric + \nabla^2 f - \mu df \otimes df = \lambda g,$$

where Ric denotes the Ricci tensor of (M^n, g) , while ∇^2 and \otimes stand for the Hessian and the tensorial product, respectively.

As a particular case of (1.1) we shall consider the following.

Definition 1. We say that (M^n, g) is a generalized m -quasi-Einstein metric if there exist two smooth functions f and λ on M satisfying

$$(1.2) \quad Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g,$$

where $0 < m \leq \infty$ is an integer. The tensor $Ric_f = Ric + \nabla^2 f - \frac{1}{m} df \otimes df$ is called *Bakry-Emery Ricci tensor*.

In particular, we have

$$(1.3) \quad Ric(\nabla f, \nabla f) + \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle = \frac{1}{m} |\nabla f|^4 + \lambda |\nabla f|^2,$$

where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ stand for the metric g and its associated norm, respectively.

Date: May 29, 2012.

2000 Mathematics Subject Classification. Primary 53C25, 53C20, 53C21; Secondary 53C65.

Key words and phrases. Ricci Soliton, quasi-Einstein metrics, Bakry-Emery Ricci tensor, scalar curvature.

^{1,2} Partially supported by CNPq-BR.

Moreover, if R stands for the scalar curvature of (M^n, g) , then, taking trace of both members of equation (1.2) we deduce

$$(1.4) \quad R + \Delta f - \frac{1}{m}|\nabla f|^2 = \lambda n.$$

Thereby we derive

$$(1.5) \quad \langle \nabla f, \nabla R \rangle + \langle \nabla f, \nabla \Delta f \rangle = \frac{1}{m} \langle \nabla f, \nabla |\nabla f|^2 \rangle + n \langle \nabla \lambda, \nabla f \rangle.$$

One notices that combining equations (1.2) and (1.4) we infer

$$(1.6) \quad \nabla^2 f - \frac{\Delta f}{n} g = \frac{1}{m} (df \otimes df - \frac{1}{n} |\nabla f|^2 g) - (Ric - \frac{R}{n} g).$$

It is important to point out that if $m = \infty$ and λ is constant, equation (1.2) reduces to one associated to a gradient Ricci soliton, for a good survey in this subject we recommend the work due to Cao in [8], as well as if λ is only constant and m is a positive integer, it corresponds to m -quasi-Einstein metrics that are exactly those n -dimensional manifolds which are the base of an $(n + m)$ -dimensional Einstein warped product, for more details see [9], [10], [14] and [5]. The 1-quasi-Einstein metrics satisfying $\Delta e^{-f} + \lambda e^{-f} = 0$ are more commonly called *static metrics*, for more details see [12]. Static metrics have been studied extensively for their connection to scalar curvature, the positive mass theorem and general relativity, see e.g. [1], [2] and [12]. In [14] it was given some classification for m -quasi-Einstein metrics where the base has non-empty boundary. Moreover, they have proved a characterization for m -quasi-Einstein metric when the base is locally conformally flat. In addition, considering $m = \infty$ in equation (1.2) we obtain the almost Ricci soliton equation, for more details see [16] and [4]. We also point out that, Catino [11] have proved that around any regular point of f a generalized m -quasi Einstein metric $(M^n, g, \nabla f, \lambda)$ with harmonic Weyl tensor and $W(\nabla f, \dots, \nabla f) = 0$ is locally a warped product with $(n - 1)$ -dimensional Einstein fibers.

A generalized m -quasi-Einstein manifold $(M^n, g, \nabla f, \lambda)$ will be called *trivial* if the potential function f is constant. Otherwise, it will be called *nontrivial*.

We observe that the triviality definition implies that M^n is an Einstein manifold, but the converse is not true. Meanwhile, we shall show in Theorem 1 that when $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, is Einstein, but not trivial, it will be isometric to a space form with a well defined potential f . Introducing the function $u = e^{-\frac{f}{m}}$ on M we immediately have $\nabla u = -\frac{u}{m} \nabla f$, moreover the next relation, which can be found in [9], is true

$$(1.7) \quad \nabla^2 f - \frac{1}{m} df \otimes df = -\frac{m}{u} \nabla^2 u.$$

In particular, ∇u is a conformal vector field, i.e. $\frac{1}{2} \mathcal{L}_{\nabla u} g = \rho g$, for some smooth function ρ defined on M , if and only if M^n is an Einstein manifold. Hence, on a surface M^2 , ∇u is always a conformal vector field.

Before to announce our main result we present a family of nontrivial examples on a space form. Let us start with a standard sphere (\mathbb{S}^n, g_0) , where g_0 is its canonical metric.

Example 1. On the standard unit sphere (\mathbb{S}^n, g_0) , $n \geq 2$, we consider the following function

$$(1.8) \quad f = -m \ln \left(\tau - \frac{h_v}{n} \right),$$

where τ is a real parameter lying in $(1/n, +\infty)$ and h_v is some height function with respect to a fixed unit vector $v \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, here we are considering \mathbb{S}^n as a hypersurface in \mathbb{R}^{n+1} , and $h_v : \mathbb{S}^n \rightarrow \mathbb{R}$ is given by $h_v(x) = \langle x, v \rangle$. Taking into account that $\nabla^2 h_v = -h_v g_0$ and $u = e^{-\frac{f}{m}} = \tau - \frac{h_v}{n}$, we deduce from (1.7) that

$$(1.9) \quad \nabla^2 f - \frac{1}{m} df \otimes df = -m \frac{\tau - u}{u} g_0.$$

Since the Ricci tensor of (\mathbb{S}^n, g_0) is given by $\text{Ric} = (n-1)g_0$, it is enough to consider $\lambda = (n-1) - m \frac{\tau - u}{u}$ in order to build a desired non trivial such structure on (\mathbb{S}^n, g_0) .

We now present a similar example as before on the Euclidean space (\mathbb{R}^n, g_0) , where g_0 is its canonical metric.

Example 2. On the Euclidean space (\mathbb{R}^n, g_0) , $n \geq 2$, we consider the following function

$$(1.10) \quad f = -m \ln (\tau + |x|^2),$$

where τ is a positive real parameter and $|x|$ is the Euclidean norm. Taking into account that $\nabla^2 |x|^2 = 2g_0$ and $u = e^{-\frac{f}{m}} = \tau + |x|^2$, we deduce from (1.7) that

$$(1.11) \quad \nabla^2 f - \frac{1}{m} df \otimes df = -2 \frac{m}{u} g_0.$$

Since the Ricci tensor of (\mathbb{R}^n, g_0) is flat, it is enough to consider $\lambda = -2 \frac{m}{u}$ in order to obtain a desired non trivial structure on (\mathbb{R}^n, g_0) .

On the other hand, concerning to hyperbolic space we have the following.

Example 3. Regarding the hyperbolic space $\mathbb{H}^n(-1) \subset \mathbb{R}^{n,1} : \langle x, x \rangle_0 = -1, x_1 > 0$, where $\mathbb{R}^{n,1}$ is the Euclidean space \mathbb{R}^{n+1} endowed with the inner product $\langle x, x \rangle_0 = -x_1^2 + x_2^2 + \dots + x_{n+1}^2$. We now follow the argument used on \mathbb{S}^n . First, we fixe a vector $v \in \mathbb{H}^n(-1) \subset \mathbb{R}^{n,1}$ and we consider a hight function $h_v : \mathbb{H}^n(-1) \rightarrow \mathbb{R}$ given by $h_v(x) = \langle x, v \rangle_0$. In this case, we have $\nabla^2 h_v = h_v g_0$. Then, taking

$$(1.12) \quad u = e^{-\frac{f}{m}} = \tau + h_v, \tau > -1$$

we have from (1.7)

$$(1.13) \quad \nabla^2 f - \frac{1}{m} df \otimes df = -m \frac{u - \tau}{u} g_0.$$

Reasoning as in the spherical case it is enough to consider $\lambda = -(n-1) - m \frac{\tau - u}{u}$ in order to build a non trivial such structure on (\mathbb{H}^n, g_0) .

Now we announce the main theorem.

Theorem 1. Let $(M^n, g, \nabla f, \lambda)$ be a non trivial generalized m -quasi-Einstein metric with $n \geq 3$. Suppose that either (M^n, g) is an Einstein manifold or ∇u is a conformal vector field. Then one the following statements holds:

- (1) M^n is isometric to a standard sphere $\mathbb{S}^n(r)$. In particular, f is, up to constant, given by (1.8).

- (2) M^n is isometric to a Euclidean space \mathbb{R}^n . In particular, f is, up to change of coordinates, given by (1.10).
- (3) M^n is isometric to a hyperbolic space \mathbb{H}^n , provided u has only one critical point. In particular, f is, up to constant, given according to (1.12).

As a consequence of this theorem we obtain the following corollary.

Corollary 1. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, be a compact non trivial generalized m -quasi-Einstein metric such that $\int_M \text{Ric}(\nabla u, \nabla u) d\mu \geq \frac{n-1}{n} \int_M (\Delta u)^2 d\mu$, where $d\mu$ stands for the Riemannian measure associated to g . Then M^n is isometric to a standard sphere $\mathbb{S}^n(r)$. Moreover, the potential f is the same of identity (1.8).*

Before to announce the next results we point out that they are generalizations of ones found in [15] and [3] for Ricci solitons, [4] for almost Ricci solitons and [9] for quasi-Einstein metrics. First, we have the following theorem.

Theorem 2. *Let $(M^n, g, \nabla f, \lambda)$ be a compact generalized m -quasi-Einstein metric. Then M^n is trivial provided:*

- (1) $\int_M \text{Ric}(\nabla f, \nabla f) d\mu \leq \frac{2}{m} \int_M |\nabla f|^2 \Delta f d\mu - (n-2) \int_M \langle \nabla \lambda, \nabla f \rangle d\mu$.
- (2) $R \geq \lambda n$ or $R \leq \lambda n$.

Now, if $(M^n, g, \nabla f, \lambda)$ is a generalized m -quasi-Einstein metric and m is finite, we shall present conditions in order to obtain $\nabla f \equiv 0$.

Theorem 3. *Let $(M^n, g, \nabla f, \lambda)$ be a complete generalized m -quasi-Einstein metric with m finite. Then $\nabla f \equiv 0$, if one of the following conditions holds:*

- (1) M^n is non compact, $n\lambda \geq R$ and $|\nabla f| \in L^1(M^n)$. In particular, M^n is an Einstein manifold.
- (2) (M^n, g) is Einstein and ∇f is a conformal vector field.

2. PRELIMINARIES

In this section we shall present some preliminaries which will be useful for the establishment of the desired results. The first one is a general lemma for a vector field $X \in \mathfrak{X}(M^n)$ on a Riemannian manifold M^n .

Lemma 1. *Let (M^n, g) be a Riemannian manifold and $X \in \mathfrak{X}(M^n)$. Then the following statements hold:*

- (1) *If $(X^\flat \otimes X^\flat) = \rho g$ for some smooth function $\rho : M \rightarrow \mathbb{R}$, then $\rho = |X|^2 = 0$. In particular, the unique solution of the equation $df \otimes df = \rho g$ is f constant.*
- (2) *If M^n is compact and X is a conformal vector field, then $\int_M |X|^2 \text{div } X d\mu = 0$. In particular, if $X = \nabla f$ is a gradient conformal vector field, then $\int_M |\nabla f|^2 \Delta f d\mu = 0$.*

Proof. Since $(X^\flat \otimes X^\flat)$ is a degenerate $(0, 2)$ tensor the first statement is trivial. Taking into account that X is a conformal vector field we have $\frac{1}{2} \mathcal{L}_X g = \rho g$, where $\rho = \frac{1}{n} \text{div } X$. From which we obtain

$$(2.1) \quad |X|^2 \text{div } X = n \langle \nabla_X X, X \rangle.$$

On the other hand, since $\text{div}(|X|^2 X) = |X|^2 \text{div } X + 2 \langle \nabla_X X, X \rangle$, one has

$$(2.2) \quad \text{div}(|X|^2 X) = \frac{n+2}{n} |X|^2 \text{div } X,$$

which allows us to complete the proof of the lemma. \square

The following formulae from [15] will be useful: on a Riemannian manifold (M^n, g) we have

$$(2.3) \quad \operatorname{div}(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + D_X \operatorname{div} X,$$

$$(2.4) \quad \operatorname{div}(\mathcal{L}_{\nabla f} g)(Z) = 2\operatorname{Ric}(Z, \nabla f) + 2D_Z \operatorname{div} \nabla f,$$

or on $(1, 1)$ -tensorial notation

$$(2.5) \quad \operatorname{div} \nabla^2 f = \operatorname{Ric}(\nabla f) + \nabla \Delta f$$

and

$$(2.6) \quad \frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + D_{\nabla f} \operatorname{div} \nabla f + \operatorname{Ric}(\nabla f, \nabla f).$$

Taking into account that $\operatorname{div}(\lambda I)(X) = \langle \nabla \lambda, X \rangle$, where λ is a smooth function on M^n and $X \in \mathfrak{X}(M)$, equation (2.3) allows us to deduce the following lemma.

Lemma 2. *Let $(M^n, g, \nabla f, \lambda)$ be a generalized m -quasi-Einstein metric. Then we have*

- (1) $\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 - \operatorname{Ric}(\nabla f, \nabla f) + \frac{2}{m}|\nabla f|^2 \Delta f - (n-2)\langle \nabla \lambda, \nabla f \rangle.$
- (2) $\frac{1}{2}\nabla R = \frac{m-1}{m}\operatorname{Ric}(\nabla f) + \frac{1}{m}(R - (n-1)\lambda)\nabla f + (n-1)\nabla \lambda.$
- (3) $\nabla(R + |\nabla f|^2 - 2(n-1)\lambda) = 2\lambda \nabla f + \frac{2}{m}\{\nabla_{\nabla f} \nabla f + (|\nabla f|^2 - \Delta f)\nabla f\}.$

Proof. Since $\operatorname{Ric} + \nabla^2 f - \frac{1}{m}df \otimes df = \lambda g$ we use the second contracted Bianchi identity

$$(2.7) \quad \nabla R = 2\operatorname{div} \operatorname{Ric}$$

as well as the next identity

$$(2.8) \quad \operatorname{div}(df \otimes df) = \Delta f \nabla f + \nabla_{\nabla f} \nabla f$$

and (2.5) to deduce

$$(2.9) \quad \nabla R + 2\operatorname{Ric}(\nabla f) + 2\nabla \Delta f - \frac{2}{m}\Delta f \nabla f - \frac{2}{m}\nabla_{\nabla f} \nabla f = 2\nabla \lambda.$$

In particular one deduces

$$(2.10) \quad \langle \nabla R, \nabla f \rangle + 2\operatorname{Ric}(\nabla f, \nabla f) + 2\langle \nabla \Delta f, \nabla f \rangle - \frac{2}{m}\Delta f |\nabla f|^2 - \frac{2}{m}\langle \nabla_{\nabla f} \nabla f, \nabla f \rangle = 2\langle \nabla \lambda, \nabla f \rangle.$$

Next using (1.5) and (2.6) jointly with the last identity we conclude

$$(2.11) \quad \frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 - \operatorname{Ric}(\nabla f, \nabla f) + \frac{2}{m}|\nabla f|^2 \operatorname{div} \nabla f - (n-2)\langle \nabla \lambda, \nabla f \rangle,$$

which finishes the first statement of the lemma. On the other hand, substituting $\Delta f = -R + \lambda n + \frac{1}{m}|\nabla f|^2$ and remembering that $\nabla|\nabla f|^2 = 2\nabla_{\nabla f} \nabla f$ we use once more (2.9) to write

$$\begin{aligned} \frac{1}{2}\nabla R &= -\operatorname{Ric}(\nabla f) - \nabla(-R + \lambda n + \frac{1}{m}|\nabla f|^2) + \frac{1}{m}\Delta f \nabla f + \frac{1}{m}\nabla_{\nabla f} \nabla f + \nabla \lambda \\ &= -\operatorname{Ric}(\nabla f) + \nabla R - \frac{1}{m}\nabla_{\nabla f} \nabla f + \frac{1}{m}\Delta f \nabla f - (n-1)\nabla \lambda. \end{aligned}$$

Of which we deduce

$$(2.12) \quad \frac{1}{2}\nabla R = Ric(\nabla f) - \frac{1}{m}\Delta f \nabla f + \frac{1}{m}\nabla_{\nabla f} \nabla f + (n-1)\nabla \lambda.$$

We now use the fundamental equation to write

$$(2.13) \quad \nabla_{\nabla f} \nabla f = \lambda \nabla f + \frac{1}{m}|\nabla f|^2 \nabla f - Ric(\nabla f).$$

In particular, combining (2.12) and (2.13) we obtain

$$\begin{aligned} \frac{1}{2}\nabla R &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}\left(\lambda + \frac{1}{m}|\nabla f|^2 - \Delta f\right)\nabla f + (n-1)\nabla \lambda \\ &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}(R - (n-1)\lambda)\nabla f + (n-1)\nabla \lambda, \end{aligned}$$

which gives the second assertion.

Finally, noticing that $\frac{1}{2}\nabla R + \frac{1}{2}\nabla|\nabla f|^2 = \frac{1}{2}\nabla R + \nabla_{\nabla f} \nabla f$ we use the last equation and (2.13) to write

$$\begin{aligned} \frac{1}{2}\nabla R + \frac{1}{2}\nabla|\nabla f|^2 &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}(R - (n-1)\lambda)\nabla f + (n-1)\nabla \lambda \\ &\quad + \lambda \nabla f + \frac{1}{m}|\nabla f|^2 \nabla f - Ric(\nabla f) \end{aligned}$$

Thus, using equation (1.4) once more, we achieve

$$\begin{aligned} \nabla(R + |\nabla f|^2 - 2(n-1)\lambda) - 2\lambda \nabla f &= \frac{2}{m}\{(|\nabla f|^2 + R - (n-1)\lambda)\nabla f - Ric(\nabla f)\} \\ &= \frac{2}{m}\{(|\nabla f|^2 + R - n\lambda + \lambda)\nabla f - Ric(\nabla f)\} \\ &= \frac{2}{m}\{(|\nabla f|^2 + \frac{1}{m}|\nabla f|^2 - \Delta f + \lambda)\nabla f - Ric(\nabla f)\} \\ &= \frac{2}{m}\{\nabla_{\nabla f} \nabla f + (|\nabla f|^2 - \Delta f)\nabla f\}, \end{aligned}$$

which concludes the proof of the lemma. \square

It is convenient to point out that for $m = \infty$ and λ constant, assertion (3) of the last lemma is a generalization of the classical Hamilton equation [13] for a gradient Ricci soliton: $R + |\nabla f|^2 - 2\lambda f = C$, where C is constant, as well as for the following relation: $\nabla(R + |\nabla f|^2 - 2(n-1)\lambda) = 2\lambda \nabla f$, that was proved in [4] for an almost Ricci soliton. Choosing $Z \in \mathfrak{X}(M)$, we deduce from the first assertion of Lemma 2 the following identity

$$(2.14) \quad \frac{1}{2}\langle \nabla R, Z \rangle = \frac{m-1}{m}Ric(\nabla f, Z) + \frac{1}{m}(R - (n-1)\lambda)\langle \nabla f, Z \rangle + (n-1)\langle \nabla \lambda, Z \rangle.$$

We now present the main result of this section. Taking in account that $u = e^{-\frac{f}{m}}$ we have the following lemma.

Lemma 3. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, be a generalized m -quasi-Einstein metric. If, in addition M^n is Einstein, then we have*

$$(2.15) \quad \nabla^2 u = \left(-\frac{R}{n(n-1)}u + \frac{c}{m}\right)g,$$

where c is constant.

Proof. Since M^n is Einstein and $n \geq 3$ we have $Ric = \frac{R}{n}g$ with R constant. In particular, it follows from (1.7) that

$$(2.16) \quad \nabla^2 u = \frac{1}{m} \left(\frac{R}{n} u - \lambda u \right) g.$$

Whence, using (2.5) we deduce

$$(2.17) \quad Ric(\nabla u) + \nabla \Delta u = \frac{1}{m} \nabla \left(\frac{R}{n} u - \lambda u \right).$$

Therefore we infer

$$(2.18) \quad \frac{R}{n} \nabla u + \nabla \Delta u = \frac{R}{nm} \nabla u - \frac{1}{m} \nabla(\lambda u).$$

On the other hand, in accordance with (1.2) and (1.7) we deduce

$$(2.19) \quad \Delta u = \frac{R}{m} u - \frac{n}{m} \lambda u.$$

We now compare (2.18) and (2.19) to obtain

$$(2.20) \quad \nabla(\lambda u) = R \frac{(m+n-1)}{n(n-1)} \nabla u.$$

Therefore we deduce $\lambda u = R \frac{(m+n-1)}{n(n-1)} u - c$, where c is constant. Next we use this value of λu in (2.16) to complete the proof of the lemma. \square

3. PROOFS OF THE MAIN RESULTS

3.1. Proof of Theorem 1.

Proof. First of all, we notice that (1.7) gives that M^n is Einstein if and only if ∇u is a conformal vector field. Since f is not constant and we are supposing that ∇u is a non trivial conformal vector field, which enables us to write $\frac{1}{2} \mathcal{L}_{\nabla u} g = \nabla^2 u = \frac{\Delta u}{n} g$, we deduce that M^n is Einstein. Moreover, using (1.2) and (1.7) we deduce

$$Ric = \left(\lambda + m \frac{\Delta u}{nu} \right) g.$$

Since $n \geq 3$, we have from Schur's Lemma that $R = n\lambda + m \frac{\Delta u}{u}$ is constant.

On the other hand, from Lemma 3 we have

$$\nabla^2 u = \left(-\frac{R}{n(n-1)} u + \frac{c}{m} \right) g$$

where c is constant. Therefore, we are in position to apply Theorem 2 due to Tashiro [17] to deduce that M^n is a space form.

If R is positive, we may assume that M^n is isometric to a unit standard sphere \mathbb{S}^n . Since $R = n(n-1)$ we deduce from Lemma 3 that $\Delta u + nu = kn$, where k is constant. Then, up to constant, u is a first eigenfunction of the Laplacian of \mathbb{S}^n . Therefore, we have $u = h_v(x) = \langle x, v \rangle + k$, where v is a linear combination of unit vectors in \mathbb{R}^{n+1} . Hence, f is, up to constant, given by (1.8).

Next, if $R = 0$ we have from (2.20) that c is not zero. In this case M^n is isometric to a Euclidean space \mathbb{R}^n . Using once more Lemma 3 we obtain $\Delta u = k$, where k

is constant. Since u must be positive, up change of coordinates, we deduce that $u(x) = |x|^2 + \tau$, with $\tau > 0$.

Finally, if $R < 0$, it follows from Theorem 2 of [17] that M^n is isometric to a hyperbolic space, since we have only one critical point for u . Now let us suppose that M^n is isometric to $\mathbb{H}^n(-1)$. We can use the same argument due to Tashiro [17] to conclude that, up to constant, $u = h_v + \tau$, $\tau > -1$, with $v \in \mathbb{H}^n(-1)$, since in this case $\langle x, v \rangle_0 = -\cosh \eta(x, v)$, where $\eta(x, v)$ is the time-like angle between x and v , which is exactly the geodesic distance between them. Therefore, we complete the proof of the theorem. \square

3.2. Proof of Corollary 1.

Proof. On integrating Bochner's formula we obtain

$$(3.1) \quad \int_M |\nabla^2 u - \frac{\Delta u}{n} g|^2 d\mu = \frac{n-1}{n} \int_M (\Delta u)^2 d\mu - \int_M Ric(\nabla u, \nabla u) d\mu.$$

In particular, from our assumption we conclude that

$$(3.2) \quad \int_M |\nabla^2 u - \frac{\Delta u}{n} g|^2 d\mu = 0.$$

Whence, we deduce that ∇u is a non trivial conformal vector field. Then, for $n \geq 3$, we can apply Theorem 1 to conclude the proof of the corollary. \square

3.3. Proof of Theorem 2.

Proof. First we integrate the identity derived in Lemma 2 and Stokes' formula to infer

$$(3.3) \quad \int_M |\nabla^2 f|^2 d\mu = \int_M Ric(\nabla f, \nabla f) d\mu - \frac{2}{m} \int_M |\nabla f|^2 \Delta f d\mu + (n-2) \int_M \langle \nabla \lambda, \nabla f \rangle d\mu.$$

On the other hand, since we are assuming that the right hand of above identity is less than or equal to zero, we obtain $\nabla^2 f = 0$. Therefore, $\Delta f = 0$, which implies by Hopf's theorem that f is constant and we finish the establishment of the first assertion.

Proceeding one notices that for $m = \infty$, using equation (1.4) the result follows. On the other hand, for m finite, considering once more the auxiliary function $u = e^{-\frac{f}{m}}$, as we already saw $\Delta u = \frac{u}{m}(R - \lambda n)$. Since M^n is compact, $u > 0$ and $(R - n\lambda) \geq 0 (\leq 0)$, we can use once more Hopf's theorem to deduce that u is constant and so is f . From which we complete the proof of the theorem. \square

3.4. Proof of Theorem 3.

Proof. Taking into account identity (1.4) we obtain

$$(3.4) \quad m \operatorname{div} \nabla f = |\nabla f|^2 + m(n\lambda - R).$$

By one hand $m \operatorname{div} \nabla f \geq 0$, since $(n\lambda - R) \geq 0$. On the other hand, if $|\nabla f| \in L^1(M^n)$, we may invoke Proposition 1 in [7], which is a generalization of a result due to Yau [18] for subharmonic functions, to derive that $\operatorname{div} \nabla f = 0$. Next, we may use equation (3.4) to conclude that $\nabla f \equiv 0$, as well as $n\lambda = R$. Therefore, f is constant and M^n is an Einstein manifold, which gives the first assertion. Now let

us suppose that (M^n, g) is an Einstein manifold, in particular a surface has this propriety. If ∇f is a conformal vector field with conformal factor ρ , here we can have a Killing vector field, then $\nabla^2 f = \rho g$, where $\rho = \frac{1}{n} \operatorname{div} \nabla f$. Since $Ric = \frac{R}{n} g$ we deduce from equation (1.6) that

$$(3.5) \quad \frac{1}{m} (df \otimes df) = |\nabla f|^2 g.$$

But, using that m is finite, we can apply Lemma 1 to conclude that $\nabla f \equiv 0$, which completes the proof of the theorem. \square

4. INTEGRAL FORMULAE FOR GENERALIZED m -QUASI-EINSTEIN METRICS

In this section we shall introduce some integral formulae for a compact generalized m -quasi-Einstein metric. Before, we present the next result which is a natural extension of one obtained for an almost Ricci soliton in [4], as well as a similar one in [16].

Lemma 4. *Let $(M^n, g, \nabla f, \lambda)$ be a generalized m -quasi-Einstein metric. Then we have*

$$\begin{aligned} \frac{1}{2} \Delta R &= -|\nabla^2 f - \frac{\Delta f}{n} g|^2 - \left\{ \frac{m+n}{nm} \right\} (\Delta f)^2 - \frac{n}{2} \langle \nabla f, \nabla \lambda \rangle + \langle \nabla f, \nabla R \rangle \\ &\quad + \left\{ \frac{m-2}{2m} \right\} \langle \nabla f, \nabla \Delta f \rangle + \frac{1}{m} \operatorname{div} (\nabla_{\nabla f} \nabla f) + (n-1) \Delta \lambda + \lambda \Delta f. \end{aligned}$$

Proof. Initially by using assertion (3) of Lemma 2 to compute the divergence of ∇R we obtain

$$\begin{aligned} \Delta R + \Delta |\nabla f|^2 - 2(n-1) \Delta \lambda &= 2 \operatorname{div} (\lambda \nabla f) + \frac{2}{m} \left\{ \langle \nabla (|\nabla f|^2 - \Delta f), \nabla f \rangle \right. \\ &\quad \left. + (|\nabla f|^2 - \Delta f) \Delta f + \operatorname{div} (\nabla_{\nabla f} \nabla f) \right\}. \end{aligned}$$

We now use $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{1}{n} (\Delta f)^2$ with Bochner's formula to write

$$\begin{aligned} \frac{1}{2} \Delta R &= -Ric(\nabla f, \nabla f) - |\nabla^2 f - \frac{\Delta f}{n} g|^2 - \frac{1}{n} (\Delta f)^2 - \langle \nabla \Delta f, \nabla f \rangle \\ &\quad + (n-1) \Delta \lambda + \operatorname{div} (\lambda \nabla f) + \frac{2}{m} \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle \\ &\quad + \frac{1}{m} \left\{ (|\nabla f|^2 - \Delta f) \Delta f - \langle \nabla \Delta f, \nabla f \rangle + \operatorname{div} (\nabla_{\nabla f} \nabla f) \right\}. \end{aligned}$$

Next, we invoke equation (1.4) to write $\langle \nabla \Delta f, \nabla f \rangle = \langle \nabla (n\lambda + \frac{1}{m} |\nabla f|^2 - R), \nabla f \rangle$. Then the last relation becomes

$$\begin{aligned} \frac{1}{2} \Delta R &= -Ric(\nabla f, \nabla f) - |\nabla^2 f - \frac{\Delta f}{n} g|^2 - \frac{m+n}{nm} (\Delta f)^2 + (n-1) \Delta \lambda \\ &\quad - \langle \nabla (\frac{1}{m} |\nabla f|^2 - R + \lambda n), \nabla f \rangle + \frac{2}{m} \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle + \operatorname{div} (\lambda \nabla f) \\ &\quad + \frac{1}{m} \left\{ |\nabla f|^2 \Delta f - \langle \nabla \Delta f, \nabla f \rangle + \operatorname{div} (\nabla_{\nabla f} \nabla f) \right\} \\ &= -\left(Ric(\nabla f, \nabla f) + (n-1) \langle \nabla \lambda, \nabla f \rangle \right) - |\nabla^2 f - \frac{\Delta f}{n} g|^2 - \frac{m+n}{nm} (\Delta f)^2 \\ &\quad + (n-1) \Delta \lambda + \lambda \Delta f + \langle \nabla R, \nabla f \rangle \\ &\quad + \frac{1}{m} \left\{ |\nabla f|^2 \Delta f - \langle \nabla \Delta f, \nabla f \rangle + \operatorname{div} (\nabla_{\nabla f} \nabla f) \right\}. \end{aligned}$$

On the other hand, using (2.14) we can write

$$(4.1) \quad Ric(\nabla f, \nabla f) + (n-1)\langle \nabla \lambda, \nabla f \rangle = \frac{1}{2}\langle \nabla R, \nabla f \rangle + \frac{1}{m}Ric(\nabla f, \nabla f) - \frac{1}{m}(R - (n-1)\lambda)|\nabla f|^2.$$

Therefore, we compare the last two equations to obtain

$$\begin{aligned} \frac{1}{2}\Delta R &= \frac{1}{2}\langle \nabla R, \nabla f \rangle - |\nabla^2 f - \frac{\Delta f}{n}g|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta \lambda + \lambda \Delta f \\ &+ \frac{1}{m}\left\{ -Ric(\nabla f, \nabla f) + (\Delta f + R - n\lambda)|\nabla f|^2 + \lambda|\nabla f|^2 \right\} \\ &+ \frac{1}{m}\left\{ -\langle \nabla \Delta f, \nabla f \rangle + div(\nabla_{\nabla f} \nabla f) \right\} \\ &= \frac{1}{2}\langle \nabla R, \nabla f \rangle - |\nabla^2 f - \frac{\Delta f}{n}g|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta \lambda + \lambda \Delta f \\ &+ \frac{1}{m}\left\{ \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle - \langle \nabla \Delta f, \nabla f \rangle + div(\nabla_{\nabla f} \nabla f) \right\} \\ &= \frac{1}{2}\langle \nabla R, \nabla f \rangle - |\nabla^2 f - \frac{\Delta f}{n}g|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta \lambda + \lambda \Delta f \\ &+ \frac{1}{2}\langle \nabla R, \nabla f \rangle + \frac{1}{2}\langle \nabla f, \nabla \Delta f \rangle - \frac{n}{2}\langle \nabla \lambda, \nabla f \rangle \\ &- \frac{1}{m}\langle \nabla \Delta f, \nabla f \rangle + \frac{1}{m}div(\nabla_{\nabla f} \nabla f). \end{aligned}$$

We now group terms to arrive at the desired result, hence we complete the proof of the lemma. \square

As a consequence of this lemma we obtain the following integral formulae.

Theorem 4. *Let $(M^n, g, \nabla f, \lambda)$ be a compact orientable generalized m -quasi-Einstein metric. Then we have.*

- (1) $\int_M |\nabla^2 f - \frac{\Delta f}{n}g|^2 d\mu + \frac{n+2}{2n} \int_M (\Delta f)^2 d\mu = \int_M \langle \nabla f, \nabla R \rangle d\mu - \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu.$
- (2) $\int_M (Ric(\nabla f, \nabla f) + \langle \nabla f, \nabla R \rangle) d\mu = \frac{3}{2} \int_M (\Delta f)^2 d\mu + \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu.$
- (3) M^n is trivial, provided $\int_M \langle \nabla R, \nabla f \rangle d\mu \leq \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu.$
- (4) $\int_M |\nabla^2 f - \frac{\Delta f}{n}g|^2 d\mu = \frac{n-2}{2n} \int_M \langle \nabla f, \nabla R \rangle d\mu - \frac{n+2}{2nm} \int_M |\nabla f|^2 \Delta f d\mu.$

Proof. Since M^n is compact we use Lemma 4 and Stokes' formula to infer

$$\begin{aligned} \int_M |\nabla^2 f - \frac{\Delta f}{n}g|^2 d\mu &= -\left(\frac{m+n}{nm}\right) \int_M (\Delta f)^2 d\mu - \left(\frac{m-2}{2m}\right) \int_M (\Delta f)^2 d\mu \\ &- \frac{n}{2} \int_M \langle \nabla \lambda, \nabla f \rangle d\mu - \int_M \langle \nabla \lambda, \nabla f \rangle d\mu + \int_M \langle \nabla f, \nabla R \rangle d\mu. \end{aligned}$$

Therefore, we obtain

$$(4.2) \quad \int_M \left(|\nabla^2 f - \frac{\Delta f}{n}g|^2 + \frac{n+2}{2n}(\Delta f)^2 \right) d\mu = \int_M \langle \nabla f, \nabla R \rangle d\mu - \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu,$$

which gives the first statement.

Next, we integrate Bochner's formula to get

$$(4.3) \quad \int_M Ric(\nabla f, \nabla f) d\mu + \int_M |\nabla^2 f|^2 d\mu + \int_M \langle \nabla f, \nabla \Delta f \rangle d\mu = 0.$$

Since $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = \int_M |\nabla^2 f|^2 d\mu - \frac{1}{n} \int_M (\Delta f)^2 d\mu$ we use Stokes' formula once more to deduce

$$(4.4) \quad \int_M Ric(\nabla f, \nabla f) d\mu + \int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = \frac{n-1}{n} \int_M (\Delta f)^2 d\mu.$$

Now, comparing (4.2) with (4.4) we obtain

$$\int_M (Ric(\nabla f, \nabla f) + \langle \nabla f, \nabla R \rangle) d\mu = \frac{3}{2} \int_M (\Delta f)^2 d\mu + \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu,$$

that was to be proved.

On the other hand, if $\int_M \langle \nabla R, \nabla f \rangle d\mu \leq \frac{n+2}{2} \int_M \langle \nabla f, \nabla \lambda \rangle d\mu$, in particular this occurs if R and λ are both constant, we deduce from the first assertion

$$(4.5) \quad \int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu + \frac{n+2}{2n} \int_M (\Delta f)^2 d\mu = 0,$$

which implies that f must be constant, so M^n is trivial.

Finally, from (1.4) we can write $\int_M \langle \nabla f, \nabla \lambda \rangle d\mu = \frac{1}{n} \int_M \langle \nabla f, \nabla (R + \Delta f - \frac{1}{m} |\nabla f|^2) \rangle d\mu$. Hence, by using equation (4.2) we infer

$$\begin{aligned} \int_M \left(|\nabla^2 f - \frac{\Delta f}{n} g|^2 + \frac{n+2}{2n} (\Delta f)^2 \right) d\mu &= \frac{n-2}{2n} \int_M \langle \nabla f, \nabla R \rangle d\mu + \frac{n+2}{2n} \int_M (\Delta f)^2 d\mu \\ &+ \frac{n+2}{2nm} \int_M \langle \nabla f, \nabla |\nabla f|^2 \rangle d\mu. \end{aligned}$$

Therefore, after cancelations and using Stokes' formula, we deduce

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = \frac{n-2}{2n} \int_M \langle \nabla f, \nabla R \rangle d\mu - \frac{n+2}{2nm} \int_M |\nabla f|^2 \Delta f d\mu,$$

which completes the proof of the theorem. \square

Now we remember that for a conformal vector field X on a compact Riemannian manifold M^n we have $\int_M \mathcal{L}_X R d\mu = \int_M \langle X, \nabla R \rangle d\mu = 0$, see e.g. [6]. On the other hand, from Lemma 1 we also have $\int_M |X|^2 \operatorname{div} X d\mu = 0$. Hence, using the last item of the above theorem we deduce that the converse of those two results are true for a gradient vector field. More exactly, we have the following corollary.

Corollary 2. *Let $(M^n, g, \nabla f, \lambda)$ be a compact orientable generalized m -quasi-Einstein metric with m finite. Then we have.*

- (1) *If $n \geq 3$, $\int_M \langle \nabla f, \nabla R \rangle d\mu = 0$ and $\int_M |\nabla f|^2 \Delta f d\mu = 0$, then ∇f is a conformal vector field.*
- (2) *If $n = 2$ and $\int_M |\nabla f|^2 \Delta f d\mu = 0$, then f is constant.*

Proof. For the first statement we use the last item of Theorem 4 to deduce $\nabla^2 f = \frac{\Delta f}{n} g$, which gives that ∇f is conformal. Next, we notice that for $n = 2$, it is enough to suppose $\int_M |\nabla f|^2 \Delta f d\mu = 0$ to conclude that ∇f is conformal. But, using Theorem 3 we conclude that f is constant, which completes the proof of the corollary. \square

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